Transformation Theory Applied to Linear Active and/or Nonbilateral Networks*

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Introduction

LINEAR transformation of the variables characterizing a dynamic system is a well-known artifice frequently used to achieve, for the pertinent equilibrium equations, a particular form which places in evidence certain fundamental properties of that system. It was effectively used in the study of electrical networks by Lord Rayleigh[1] whose particular interest lay in exploiting the advantages resulting from transformation to normal coordinates. Literally as well as conceptually, I have long found such interpretations to be extremely helpful in providing insight into both the analysis and synthesis aspects of network theory. In analysis, we arrive in this way at results that are interesting with regard to properties of the steady-state and transient response of networks;[2] in synthesis, we recognize certain canonic forms (such as Foster's forms for reactances) as physical embodiments in which the pertinent normal coordinates become a reality.[3]

Transformation theory more generally was first suggested by Cauer[4] in network synthesis as a means for obtaining equivalent networks. These transformations, which have the property of keeping selected driving-point and transfer impedances invariant, were later studied in detail by Howitt.[5] In the practical application of this procedure it was found to be lacking in usefulness because of the difficulty in establishing a correspondence between parameter matrices and network topology. In going from a network schematic to a set of parameter matrices, one must first decide upon a choice of variables. Hence, there is no fixed correspondence between a set of parameter matrices and a network structure unless we freeze upon a particular way of defining variables, which we do not wish to do because of the concomitant limitations that are thus implied.

It was in the course of these studies that I first observed that symmetry of parameter matrices for passive bilateral networks has nothing to do fundamentally with either their passive or their bilateral character (as was generally thought to be the case at that time), but rather that this symmetry was the result of the particular choice customarily made for the pertinent variables. A deviation from this custom, it was observed, led to dissymmetrical parameter and impedance matrices even though the pertinent network was both passive and bilateral.

It thus became clear that if passive bilateral networks could lead to dissymmetrical impedance and admittance matrices, then nonpassive, nonbilateral networks could be characterized by symmetrical impedance matrices if we cared to achieve such a result. A linear transformation of the variables is all that stands between symmetry and nonsymmetry in the parameter or impedance matrices, regardless of whether the pertinent network is passive and/or bilateral or neither.

It was but a short step from this result to the conclusion that a nonpassive and/or nonbilateral (linear) network could be represented analytically as a passive bilateral one in which the variables are subjected to an appropriate linear transformation.[6] Although this concept potentially provides an effective means for dealing with both the analysis and synthesis of such networks, it has received little, if any, attention. In the case of synthesis, the reason probably is the fact that the construction of a network schematic from given parameter matrices presents difficulties which, so far, at least, have not been overcome. Nevertheless, the method does yield a valuable insight into certain aspects of the synthesis problem, and for this reason a brief presentation of its essential features is given here.

Moreover, the idea that the presence of active and/or nonbilateral elements in a linear network can effectively be accounted for by an appropriate transformation of the variables, presents as many useful possibilities in analysis as it does in synthesis, and seems to have been almost totally ignored. Thus we find such circuits at present almost exclusively treated by the "dependent source" method. While the active character is in this way accountable, the possible nonbilateral property is not. Transformation theory, on the other hand, offers not only a very general and elegant way of dealing with both the nonbilateral as well as the active character of devices, but, through the existence of a considerable number of interpretive mechanisms such as geometric visualization in multidimensional spaces, and the concept of normal coordinate transformations, it represents a relatively untapped storehouse of manipulative artifices of one sort or another that

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[1] Lord Rayleigh, "The Theory of Sound," Dover Publications, Inc., New York, N. Y., vol. 1, ch. 5; 1945.

[2] E. A. Guillemin, "Making normal coordinates coincide with the meshes of an electrical network," Proc. IRE, vol. 15, pp. 935–945; November, 1927.

[3] E. A. Guillemin, "Synthesis of Passive Networks," John Wiley and Sons, New York, N. Y., ch. 3, art. 7; 1957.

[4] See W. Cauer, "Vierpole," *Elect. Nach. Tech.*, vol. 6, p. 272; July, 1929.

[5] N. Howitt, "Group theory and the electric circuit," *Phys. Rev.*, vol. 37, pp. 1583–1595; December, 1931.

[6] First pointed out by the writer in a discussion entitled "An interpretation of network relations when active unilateral elements are included with the usual passive ones," presented at the Fifth Annual Informal Conference on Network Analysis and Synthesis, sponsored by AIEE Communication Group, New York, N. Y., January 22, 1940.

may ultimately provide a clearer understanding of this broader class of linear networks.

Normal Coordinate Transformation for an Active RC Network

As has been restated recently by Linvill,[7] one of the things that a network containing active elements can provide is a transfer function with poles and zeros anywhere in the complex frequency plane even though only two kinds of elements (R and L or R and C) are involved. This situation is primarily of interest because it permits the synthesis, in terms of R's and C's and an active device (vacuum tube or transistor), of a transfer function whose poles are not confined to the negative real axis, but may (as with RLC functions) lie anywhere in the left half-plane. A rather general way of appreciating the truth of this statement is through an argument that effectively begins with a normal coordinate transformation.

Visualize a general RC network characterized in terms of node pairs and node-pair voltage variables with one vacuum tube or transistor embedded in such a way that its input and output terminal pairs become identified with node pairs 1 and 2. There are n independent node pairs altogether. Because of the embedded device, the circuit may be active and nonbilateral although its capacitance matrix is PB (passive and bilateral) and its conductance matrix, except for the elements on the intersections of its first two rows and columns, is likewise PB.

In these discussions, incidentally, a matrix will be referred to as bilateral if it is symmetrical, and passive if it is the matrix of a positive definite (or semidefinite) quadratic form. It will be spoken of as non-P if its quadratic form is not positive, and as non-B if it is not symmetrical about its principal diagonal.

Suppose now that we wish to transform to the normal coordinates of this active nonbilateral network. We do this by first applying a congruent transformation to the G (conductance) and C (capacitance) matrices that carry C into its canonic form. To start, let us assume that C has the rank n; then the canonic form of C is a unit matrix of the same rank. We may imagine this transformation of the original C matrix carried out through a succession of elementary transformations involving addition of the α-multiplied elements of one row to the respective ones of another row (and corresponding operations on columns); and if we thus add to the elements of rows 1 and 2 (columns 1 and 2, respectively) but never the reverse, then the corresponding transformation of the G matrix leaves it PB except for the 2×2 submatrix in its upper left-hand corner. In other words, although the G matrix is affected by the congruent transformation that changes C into its canonic form, it is not changed in form as to the portion that is passive and symmetrical and that portion which contributes to the possible nonpassive character of the network.

[7] J. G. Linvill, "Synthesis techniques and active networks," 1957 IRE National Convention Record, part 2, pp. 90–95.

If G were PB, the next step in the normal coordinate transformation would be to make an orthogonal transformation of G (corresponding to a rotation of coordinate axes) so as to change G to a diagonal form (cause the coordinate axes to coincide with the principal axes of the associated quadric surface) while C remains in its canonic form because its pertinent quadric surface is a sphere and hence its principal axes have no fixed orientations. Before we can discuss the analogous transformation in our present situation, it is necessary to manipulate the non-P and non-B portion of G as follows.

Let us write the 2×2 submatrix in the upper left-hand corner of G in the form

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} g_{11} & g_s \\ g_s & g_{22} \end{bmatrix} + \begin{bmatrix} 0 & -g \\ g & 0 \end{bmatrix} \tag{1}$$

in which the terms on the right are respectively the symmetric and skew-symmetric components of this submatrix. Incidentally, the device with this conductance matrix is nonpassive (active) if the symmetric component is the matrix of a nonpositive quadratic form, and it is nonbilateral if the skew-symmetric component is nonzero. Although practical devices are usually nonbilateral when they are nonpassive, these two characteristics do not necessarily occur together and should not be confused.

The total G matrix with the embedded submatrix 1 is symmetric except for the additive skew-symmetric term in (1). If we proceed with our normal coordinate transformation, ignoring this skew-symmetric term, and remind ourselves that a congruent transformation of the first term in (1) to a diagonal form leaves the skew-symmetric term unchanged, we recognize that our total G matrix will now assume the form

$$\begin{bmatrix} -g_1 & -g & 0 & 0 & \cdots \\ g & g_2 & 0 & 0 & \cdots \\ 0 & 0 & g_3 & 0 & \cdots \\ 0 & 0 & 0 & g_4 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \end{bmatrix} \tag{2}$$

while the associated C matrix still looks like a unit matrix of order n. The minus sign in the 1, 1 element in (2) arises from the assumed active character of the device having the conductance matrix 1, and the minus sign in the 1, 2 element (as well as its nonzero value) results from the nonbilateral character of this device.

If we assume that the admittance matrix of our original RC network with embedded vacuum tube or transistor is denoted by $Y(s)$, then we may summarize the transformations so far into a single congruent transformation indicated by

$$AYA_t = \begin{bmatrix} (-g_1 + s) & -g & 0 & \cdots \\ g & (g_2 + s) & 0 & \cdots \\ 0 & 0 & (g_3 + s) & \cdots \\ \cdots & \cdots & \cdots & \end{bmatrix} \tag{3}$$

in which A_t is the transpose of A and s is the complex frequency variable. Because of the assumed nonbilateral character of the embedded device, our normal coordinate transformation is not yet achieved at this stage. In fact it cannot be achieved with a real transformation unless $g_1g_2 - g^2 \geq 0$ because the simultaneous reduction of two quadratic forms to sums of squares requires that at least one of them be positive definite, and that both have symmetrical matrices.

The latter requirement we can meet through premultiplication of (3) by the matrix

$$\delta = \begin{bmatrix} -1 & 0 & 0 \cdots \\ 0 & 1 & 0 \cdots \\ 0 & 0 & 1 \cdots \\ \cdots & & \end{bmatrix} \tag{4}$$

which is the same as a unit matrix of order n except that the (1, 1) element is negative. The resulting G matrix is now symmetrical, but the canonic C matrix has the form of δ in (4) and hence its associated quadric surface is no longer a sphere. The reduction to normal coordinates cannot be completed with an orthogonal transformation. Instead we must now transform G to a canonic form; but the latter will have all positive coefficients only if $g_1g_2 - g^2 \geq 0$. In this event, a further orthogonal transformation regains a diagonal form for the C matrix; and we can then put it back into canonic form and write

$$t\delta A Y A_t t_t = \begin{bmatrix} (g_1 - s) & 0 & 0 & \cdots \\ 0 & (g_2 + s) & 0 & \cdots \\ 0 & 0 & (g_3 + s) & \cdots \\ \cdots & & & \end{bmatrix} = Y_d \tag{5}$$

in which the matrix t and its transpose t_t combine the several final transformations into a single one. The diagonal elements $g_1, g_2 \cdots$ in the final G matrix are in general not the same as the similarly denoted quantities in the matrix (3), but we shall not bother to change the notation on this account.

The matrices A, δ, t are all real and nonsingular, and the desired transformation to normal coordinates is achieved. Y_d is the resultant diagonal matrix. Its number one coordinate is unstable since it is characterized by a natural frequency on the positive real axis.

In the event that $g_1g_2 - g^2$ in (3) is negative, the transformation must lose its real character. The matrix (3), premultiplied by δ (4) and transformed congruently by

$$\theta = \begin{bmatrix} j & 0 & 0 \cdots \\ 0 & 1 & 0 \cdots \\ 0 & 0 & 1 \cdots \\ \cdots & & \end{bmatrix} \tag{6}$$

has a canonic C matrix with all positive coefficients and a symmetrical G matrix although its nondiagonal terms contain the factor $j = \sqrt{-1}$. A further orthogonal transformation (also nonreal) may be found in the usual manner to complete the transformation to normal coordinates. The end result can be put into a form analogous to (5), namely

$$t\theta\delta A Y A_t \theta_t t_t = \begin{bmatrix} (g_1 + s) & 0 & 0 & \cdots \\ 0 & (g_2 + s) & 0 & \cdots \\ 0 & 0 & (g_3 + s) & \cdots \\ \cdots & & & \end{bmatrix} = Y_d. \tag{7}$$

The elements g_1 and g_2 are now in general complex and so are the elements in the transformation matrix t.

If we assume that the given RC network with embedded vacuum tube or transistor does not have a capacitance across terminal pair 1 (which may be either the input or output of the vacuum tube or transistor) then the C matrix has rank n-1, and the 1, 1 term in its canonic form is zero. Instead of (3) we then have

$$A Y A_t = \begin{bmatrix} -g_1 & -g & 0 & \cdots \\ g & (g_2 + s) & 0 & \cdots \\ 0 & 0 & (g_3 + s) & \cdots \\ \cdots & & & \end{bmatrix}. \tag{8}$$

Premultiplication by δ, (4), then has no effect upon the C matrix, and an orthogonal transformation with an appropriate matrix t completes the conversion to normal coordinates. Instead of (5), we have in this case

$$t\delta A Y A_t t_t = \begin{bmatrix} g_1 & 0 & 0 & \cdots \\ 0 & (g_2 + s) & 0 & \cdots \\ 0 & 0 & (g_3 + s) & \cdots \\ \cdots & & & \end{bmatrix} = Y_d \tag{9}$$

where all g's are real and positive and again g_1 and g_2 do not necessarily have the same values as in (8), but this is unimportant.

The total network in this last situation is not quite as general as in the preceding cases, but there is no difficulty in always transforming straightforwardly to normal coordinates by means of a real nonsingular transformation, and, as we shall now show, the essential response characteristics obtainable are no less general.

Impedance of an Active RC Network Expressed as a Linear Combination of the Impedances of a Passive Reference Network

The RC network with embedded vacuum tube or transistor, characterized by the matrix Y, has the equilibrium equations

$$YE = I \tag{10}$$

where E is a column matrix for the node-pair voltage

variables and I is a column matrix for the current sources feeding these node pairs.

From (9)[and analogously from (5) or (7) in the other situations] we can express the matrix Y in terms of the diagonal matrix Y_d (characterizing the normal coordinates of our network) as follows:

$$Y = A^{-1}\delta t^{-1} Y_d t_t^{-1} A_t^{-1} \tag{11}$$

which we can also write

$$Y = \tau A^{-1} t^{-1} Y_d t_t^{-1} A_t^{-1} \tag{12}$$

where τ is a real nonsingular matrix given by

$$\tau = A^{-1}\delta A \ . \tag{13}$$

Since δ is its own inverse, so is the matrix τ. The collineatory transformation (13) in terms of a matrix δ, differing from the unit matrix only in that some of its diagonal elements are negative, yields a general method for the construction of a matrix that is selfinverse.[8]

The admittance matrix

$$\hat{Y} = A^{-1} t^{-1} Y_d t_t^{-1} A_t^{-1} \tag{14}$$

for Y_d as given by (9) evidently characterizes a linear passive bilateral RC network. By means of (12) and (14),

$$Y = \tau\hat{Y} \text{ or } \tau Y = \hat{Y}. \tag{15}$$

We can therefore write for equilibrium (10)

$$\tau Y E = \hat{Y} E = \hat{I} \tag{16}$$

where

$$\hat{I} = \tau I \quad \text{or} \quad I = \tau\hat{I}. \tag{17}$$

Eq. (16) expresses the equilibrium of a passive bilateral RC network (a sort of reference network), and the nonpassive nonbilateral network with the matrix Y is related to this reference network through the real nonsingular transformation of variables expressed by (17). The statement made initially, namely that an active, nonbilateral system is equivalent to a passive bilateral one, plus a linear transformation of the pertinent variables, is thus established.

The procedure leading to this result can be varied in several ways if desired. We can postmultiply (8) by δ and then transform congruently with an appropriate matrix to get, instead of (9)

$$tAYA_t\delta t_t = \begin{bmatrix} g_1 & 0 & 0 & \cdots \\ 0 & (g_2+s) & 0 & \cdots \\ 0 & 0 & (g_3+s) & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} = Y_d \ . \tag{18}$$

The same definition of τ in (13) now yields

$$Y = \hat{Y}\tau_t \quad \text{or} \quad Y\tau_t = \hat{Y} \tag{19}$$

[8] The square root of the unit matrix (which is thus seen to have an infinite number of values) is evidently given by τ. Cube roots of the unit matrix and any others are analogously constructible.

and equilibrium (10) is expressed by

$$Y\tau_t\hat{E} = \hat{Y}\hat{E} = I \tag{20}$$

where

$$\hat{E} = \tau_t E \quad \text{or} \quad E = \tau_t\hat{E} \ . \tag{21}$$

In this case the voltages rather than the currents are transformed. We can evidently also use complex transformations involving the matrix θ, (6), and we can thus transform either currents or voltages, or both. Until a particular application indicates a preference in this regard, however, there is no point in elaborating upon details.

In terms of the results expressed by (14) and (15), we can now place in evidence the results obtainable from our active, nonbilateral network in contrast with the passive bilateral reference network. The open-circuit driving-point and transfer impedances with respect to any selected node pairs are the elements of the inverse of Y. If we denote this impedance matrix by Z and that which is the inverse of $\hat{Y}$ (the impedance matrix of the passive bilateral reference network) by $\hat{Z}$, then we have from (15)

$$Z = \hat{Z}\tau \tag{22}$$

or for the elements of these matrices

$$z_{ik} = \sum_{j=1}^{n} \hat{z}_{ij}\tau_{jk} \ . \tag{23}$$

If we denote the elements of the matrix A by a_{ik} and those of its inverse by b_{ik} then in view of (13) for τ we find

$$\tau = u - 2\begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix} \times [a_{11} \neq a_{12} \cdots a_{1n}] \tag{24}$$

in which u is the unit matrix of order n and (23) can be written

$$z_{ik} = \hat{z}_{ik} - 2\sum_{j=1}^{n} \hat{z}_{ij} b_{j1} a_{1k} \ . \tag{25}$$

These results state that the impedances of the active, nonbilateral network are linear combinations of those of the reference network. Although the impedances z_{ik} have the same poles as $\hat{z}_{ik}$, they can obviously have zeros anywhere. So far as transfer impedances are concerned, this result offers nothing new, but a driving-point impedance with zeros anywhere presents a departure from results obtainable without an active element.

In a voltage transfer ratio like

$$\frac{E_k}{E_j} = \frac{z_{jk}}{z_{jj}} \ , \tag{26}$$

the common negative real poles of the z_{ik} cancel and the ratio itself therefore can have poles and zeros anywhere. If the output voltage E_k appears across a resistance, then the current through the latter is proportional to E_k and

we have a transfer admittance I_k/E_j with zeros and poles anywhere. Since we can carry through the dual to all preceding arguments and manipulations, it follows that the RC circuit with an active device can likewise potentially produce transfer impedance functions with zeros and poles located anywhere in the complex frequency plane.

These conclusions are, of course, well-known, but the above method of obtaining them is not only basically simpler and more direct, but yields an insight into the mechanism by which certain desired functions might be constructible. Thus, if we invert (14), we get for the impedance matrix of the passive bilateral reference network

$$\hat{Z} = A_t t_t Y_d^{-1} t A \tag{27}$$

in which, from (18),

$$Y_d^{-1} = \begin{bmatrix} g_1^{-1} & 0 & 0 & \cdots \\ 0 & (g_2 + s)^{-1} & 0 & \cdots \\ 0 & 0 & (g_3 + s)^{-1} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}. \tag{28}$$

The diagonal terms in Y_d^{-1} have the form of terms in a partial fraction expansion for an RC impedance. Since $\hat{Z}$ according to (27) is formed by making linear combinations of the rows and columns in (28), we see very clearly how the partial fraction expansions of the $\hat{z}_{ik}$ are constructed. In fact we can make use of the method of formation made evident by (27) and (28) to derive independently the properties of passive RC driving-point and transfer impedances.

In a synthesis procedure we are free to choose the linear row and column combinations fixing the matrices A and t, and hence also the combinations implied by the matrix τ in (23) and (25) for the impedance functions of the network with an embedded active element. The possibilities offered by this approach to active network synthesis, which are presently being investigated, will be discussed (we hope) in the not too distant future.

Although these ideas have been presented here only with respect to RC networks, their application to RL or RLC situations will follow analogous lines, and further discussion can therefore be omitted since the main objective in this paper is primarily to formulate a method of interpretation. Many details regarding its implementation still remain to be worked out.

Synthesis of a Multistage Transistor Amplifier Through Use of a Complex Linear Transformation

The possible occurrence of transformation matrices with complex elements and even some with elements that are functions of the complex frequency s may further be elaborated upon by showing how we can readily formulate a synthesis procedure for the design of a multistage transistor amplifier with prescribed frequency response using such transformations. With vacuum tubes, the design of a multistage amplifier for prescribed frequency response is ordinarily considered straightforward in view of the isolation properties of the tube which permits the over-all response function to be expressed as a product of the response functions of the separate stages. Since the transistor does not similarly isolate, the synthesis of a cascade of stages for a prescribed resultant characteristic presents a more difficult problem.

If we express the multistage transistor amplifier as a multistage vacuum tube amplifier plus a linear transformation of variables, then we can obtain the design of the former from that of the latter in the following way.

Consider a three-stage vacuum tube amplifier having the following admittance matrix

$$Y_{vt} = \begin{bmatrix} y_1 & 0 & 0 & 0 \\ g_{m1} & y_2 & 0 & 0 \\ 0 & g_{m2} & y_3 & 0 \\ 0 & 0 & g_{m3} & y_4 \end{bmatrix}. \tag{29}$$

The plate of each vacuum tube is assumed directly connected to the grid of the succeeding one; y_1 is the admittance of a circuit connected from grid to cathode of the first tube, y_2 is that of a circuit between grid and cathode of the second (or plate and cathode of the first) tube, and so fourth, and y_4 finally is the admittance of a circuit between plate and cathode of the last tube (altogether a familiar circuit arrangement) and the g_{m1}, g_{m2}, g_{m3} are the pertinent transconductances.

An analogously constructed admittance matrix for the corresponding transistor amplifier may be assumed in the form

$$Y_{tr} = \begin{bmatrix} y_{11} & g_{12} & 0 & 0 \\ g_{21} & y_{22} & g_{23} & 0 \\ 0 & g_{32} & y_{33} & g_{34} \\ 0 & 0 & g_{43} & y_{44} \end{bmatrix} \tag{30}$$

the essential difference being that all elements above the principal diagonal are not zero as in the matrix (29). We assume that the vacuum tube amplifier has been designed in the familiar manner and hence that $y_1 \cdots y_4$ in (29) are known. The admittances $y_{11} \cdots y_{44}$ in (30) are to be found such that the over-all voltage ratio for the transistor circuit is the same (except for a multiplier) as that pertaining to the vacuum tube circuit.

If we denote by E_1, E_2, E_3, E_4 the voltages across $y_1 \cdots y_4$, respectively and by E_1', E_2', E_3', E_4' the voltages across $y_{11} \cdots y_{44}$, then the equilibrium equations for the two amplifier circuits are given by

$$Y_{vt} \times E = I \tag{31}$$

and

$$Y_{tr} \times E' = I. \tag{32}$$

The variables E and E' are assumed to be connected by a linear transformation

$$E = tE' \tag{33}$$

with the transformation matrix

$$t = \begin{bmatrix} t_{11} & t_{12} & 0 & 0 \\ 0 & t_{22} & t_{23} & 0 \\ 0 & 0 & t_{33} & t_{34} \\ 0 & 0 & 0 & t_{44} \end{bmatrix}. \tag{34}$$

Since (31)–(33) yield

$$Y_{vt} \times t = Y_{tr} \tag{35}$$

we get the specific equations

$$\begin{aligned} &y_{11} = t_{11}y_1; \quad g_{12} = t_{12}y_1 \\ &g_{21} = t_{11}g_{m1}; \quad y_{22} = t_{12}g_{m1} + t_{22}y_2; \quad g_{23} = t_{23}y_2 \\ &g_{32} = t_{22}g_{m2}; \quad y_{33} = t_{23}g_{m2} + t_{33}y_3; \quad g_{34} = t_{34}y_3 \\ &g_{43} = t_{33}g_{m3}; \quad y_{44} = t_{34}g_{m3} + t_{44}y_4. \end{aligned} \tag{36}$$

From these results we see that if we choose

$$t_{12} = g_{12}/y_1; \quad t_{23} = g_{23}/y_2; \quad t_{34} = g_{34}/y_3, \tag{37}$$

then

$$\begin{aligned} y_{22} &= t_{22}y_2 + (g_{m1}g_{12}/y_1) \\ y_{33} &= t_{33}y_3 + (g_{m2}g_{23}/y_2) \\ y_{44} &= t_{44}y_4 + (g_{m3}g_{34}/y_3). \end{aligned} \tag{38}$$

For the diagonal elements in the transformation matrix (34) we choose real numbers. The adjacent diagonal elements in this matrix, according to (37), are rational functions of s. Together with $y_{11} = t_{11}y_1$ in (36), the results (38) determine the transistor amplifier network; and the relation $E_4 = t_{44}E_4'$ obtained from (33) and (34) shows that the output voltage of that network is the same frequency function as that of the vacuum tube circuit. There are sufficient free choices available to optimize various things.

Many variations of this general theme are evidently possible. I think it is safe to say that transformation theory as an aid to analysis and synthesis alike stands at the threshold of an era of limitless possibilities in the further study of active nonbilateral networks.